Logical Foundations of Cyber-Physical Systems

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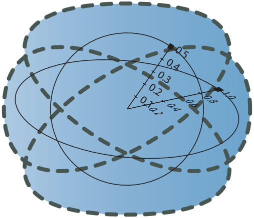
http://symbolaris.com/



DARPA

Outline

1. CPS are Multi-Dynamical Systems
   - Hybrid Systems
   - Hybrid Games
2. Dynamic Logic for Multi-Dynamical Systems
   - Syntax
   - Semantics
3. Proofs for CPS
4. Theory of CPS
   - Soundness and Completeness
   - Differential Invariants
   - Differential Radical Invariants
5. Applications
   - Ground Robots
6. Summary

Outline

Can you trust a computer to control physics?

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Rationale

1. Safety guarantees require analytic foundations
2. Foundations revolutionized digital computer science & society
3. Need even stronger foundations when software reaches out into our physical world

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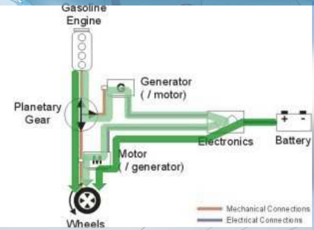
CPS Core Question

How can we provide people with cyber-physical systems they can bet their lives on?

CPS Applications



Correctness Questions in CPS Design

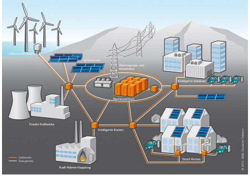
Safety The system must be safe under all circumstances

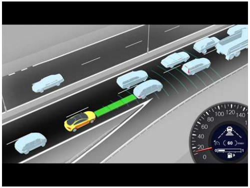
Liveness The system must reach a given goal

How do we make cyber-physical systems safe?

Extensive testing?
Code reviews?

When are we done? How many test cases are enough? Did we cover all relevant tests?





Benefits of Logical Foundations for V & V

Proving

Safety	Formalize system properties: What is "Safe"? "Reach goal"?
Models	Formalize system models
Assumptions	Make assumptions explicit
Constraints	Reveal invariants, switching conditions, starting conditions
Design	Invariants guide safe controller design
Constructive	Construct models along with their proof

Byproducts

Analyze	Determine design trade-offs & feasibility early
Synthesize	Turn high-level models into code & correctness monitors
Certify	Proofs as artifacts for certification

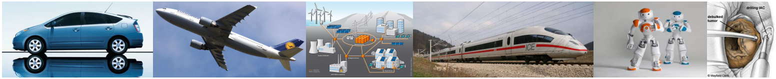
Tools

KeYmaera	Theorem prover for CPS

Successful CPS Proofs

Diverse Application Domains

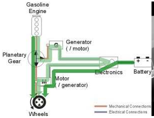
- Automotive
- Aircraft
- Railway
- Energy
- Robotics
- Surgery



Various Levels

All the way from

- Engine idle control
- Adaptive cruise control
- Highway traffic control



Successful CPS Proofs

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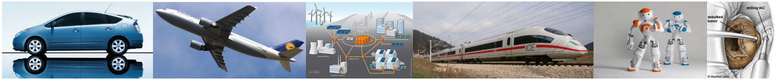

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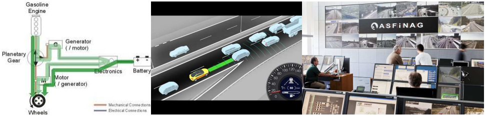
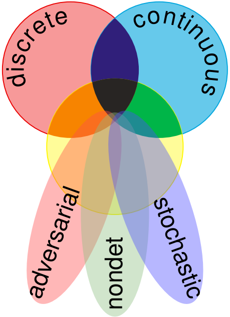


CPS are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combine multiple simple dynamical effects.

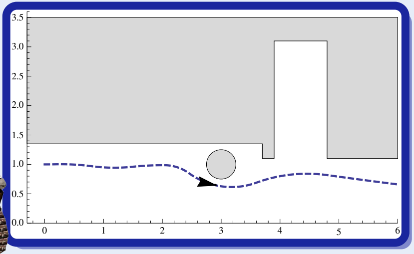
Tame Parts

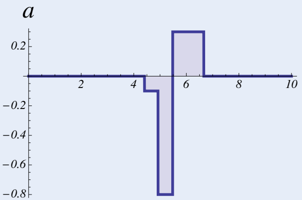
Exploiting compositionality tames complexity.

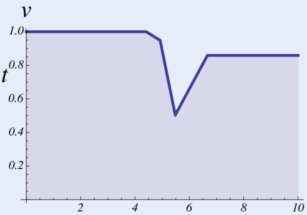
Challenge (Hybrid Systems)

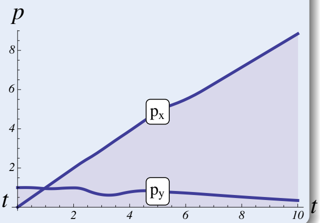
Fixed rule describing state evolution with both

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)









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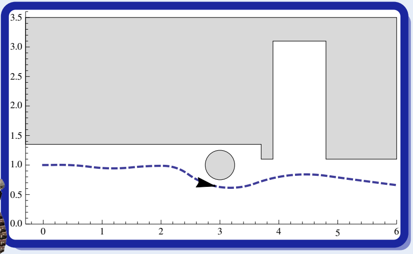

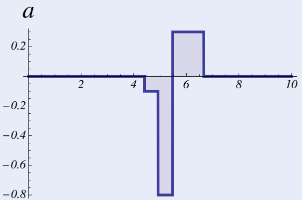

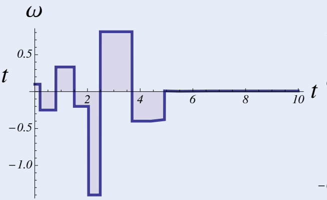

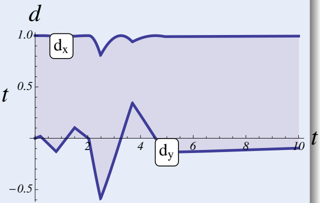

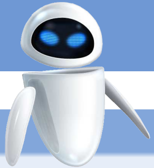

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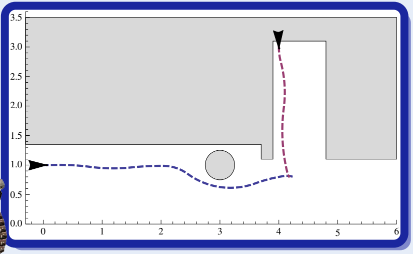

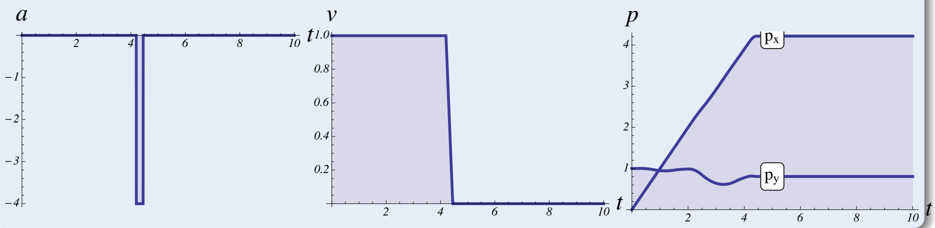

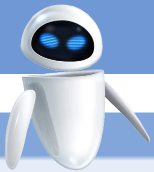

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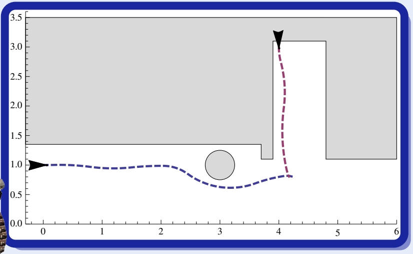
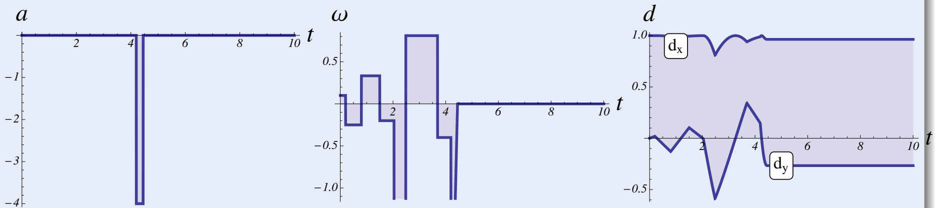
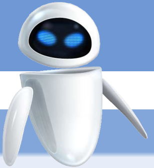


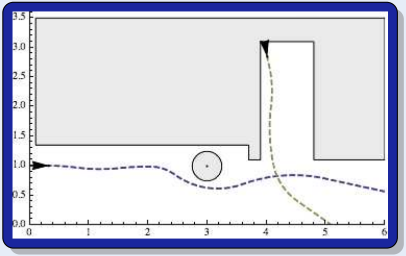


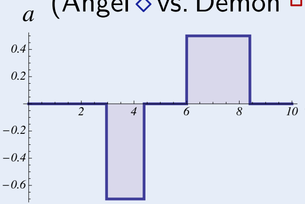


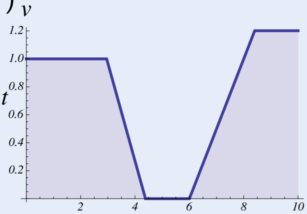
Challenge (Hybrid Games)

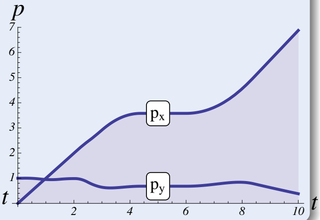
Game rules describing play evolution with

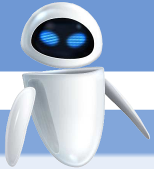
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel ◇ vs. Demon □)











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Game rules describing play evolution with

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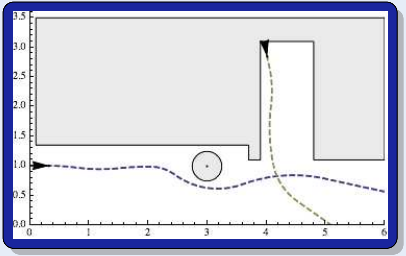
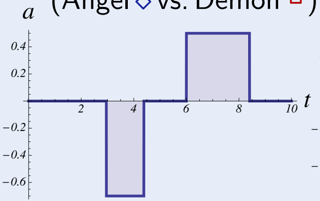
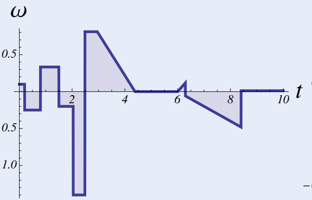
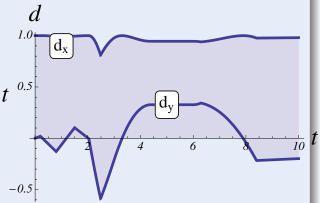
- Adversarial dynamics (Angel $\Diamond$ vs. Demon $\Box$)







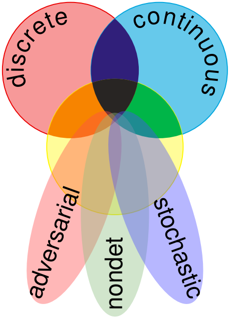


Outline

CPS are Multi-Dynamical Systems

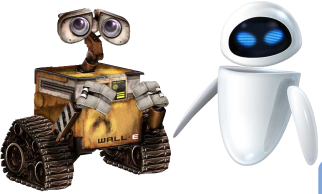
hybrid systems

HS = discrete + ODE



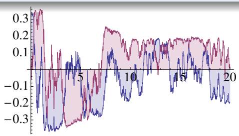
hybrid games

HG = HS + adversary



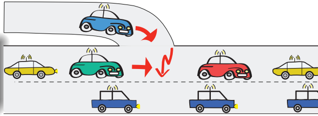
stochastic hybrid sys.

SHS = HS + stochastics

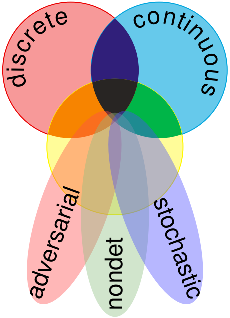


distributed hybrid sys.

DHS = HS + distributed



Family of Differential Dynamic Logics

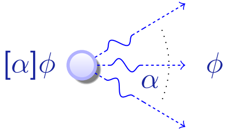


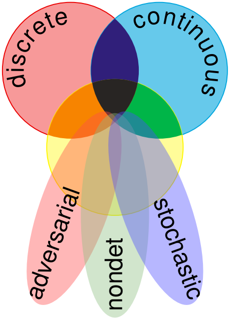
JAR'08,CADE'11,LMCS'12,LICS'12,LICS'12

Family of Differential Dynamic Logics

differential dynamic logic

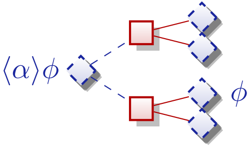
$d\mathcal{L} = \text{DL} + \text{HP}$





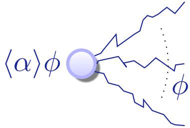
differential game logic

$dG\mathcal{L} = \text{GL} + \text{HG}$



stochastic differential DL

$Sd\mathcal{L} = \text{DL} + \text{SHP}$

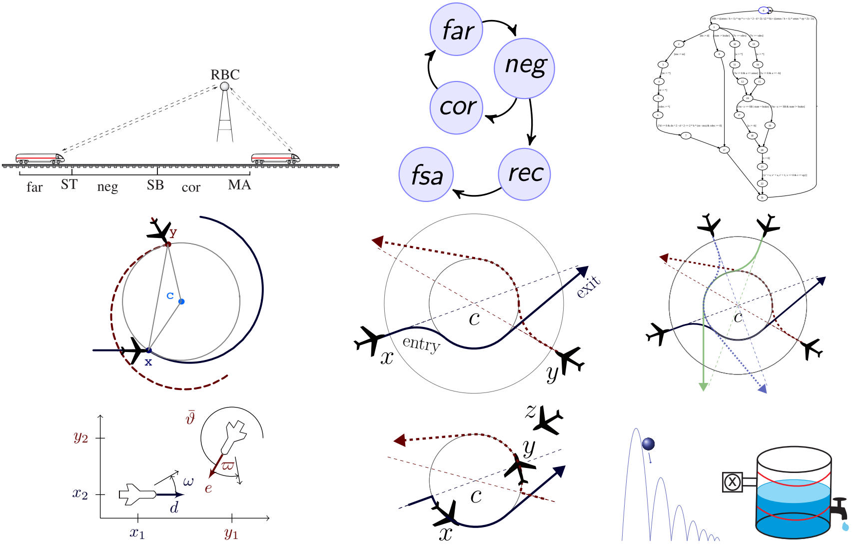


quantified differential DL

$Qd\mathcal{L} = \text{FOL} + \text{DL} + \text{QHP}$

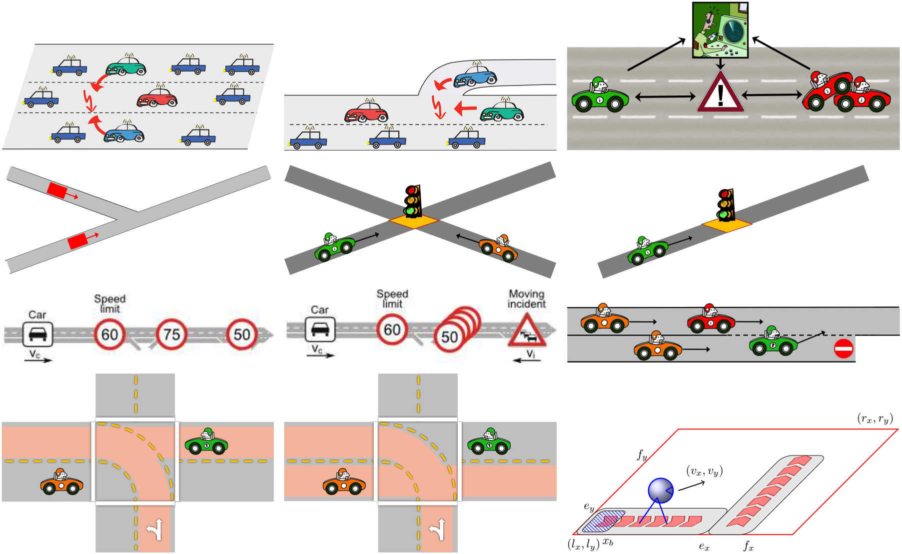
JAR'08,CADE'11,LMCS'12,LICS'12,LICS'12

Successful CPS Proofs



ICFEM'09,CAV'08,FM'09,HSCC'11

Successful CPS Proofs



FM'11,LMCS'12,ICCPS'12,ITSC'11,ITSC'13,IJCAR'12

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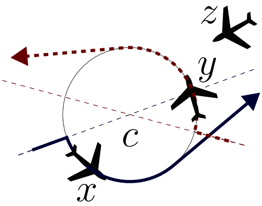

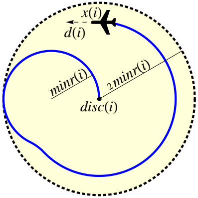

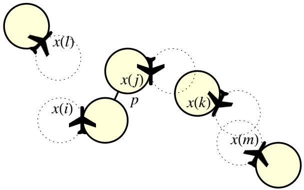

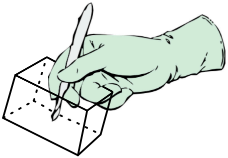

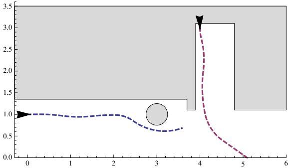

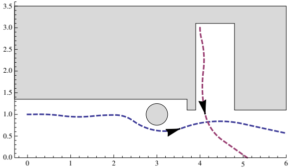

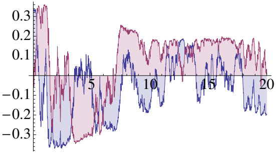

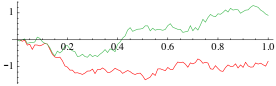

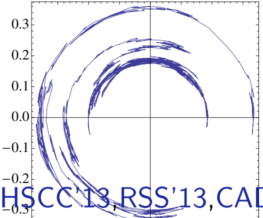

HSCC'11,HSCC'13,HSCC'13,RSS'13,CADE'12

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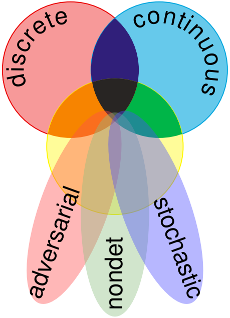

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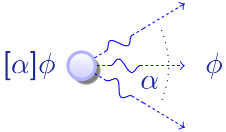
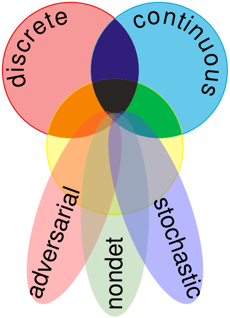
$d\mathcal{L} = \mathsf{DL} + \mathsf{HP}$





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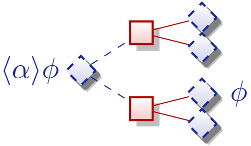

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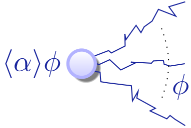

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Differential Dynamic Logic d$\mathcal{L}$: Syntax

Definition (Hybrid program α)

$$x := \theta \mid ?H \mid x' = f(x) \,\&\, H \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (d$\mathcal{L}$ Formula ϕ)

$$\theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \forall x\, \phi \mid \exists x\, \phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$

Differential Dynamic Logic dℒ: Syntax

Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

Definition (Hybrid program α)

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All Reals | Some Reals | All Runs | Some Runs

Differential Dynamic Logic d$\mathcal{L}$: Semantics

Definition (Hybrid program α)

$$
\begin{aligned}
\rho(x := \theta) &= \{(v, w) \ : \ w = v \text{ except } [\![x]\!]_w = [\![\theta]\!]_v\} \\
\rho(?H) &= \{(v, v) \ : \ v \models H\} \\
\rho(x' = f(x)) &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r\} \\
\rho(\alpha \cup \beta) &= \rho(\alpha) \cup \rho(\beta) \\
\rho(\alpha; \beta) &= \rho(\beta) \circ \rho(\alpha) \\
\rho(\alpha^*) &= \bigcup_{n \in \mathbb{N}} \rho(\alpha^n)
\end{aligned}
$$

Definition (d$\mathcal{L}$ Formula ϕ)

$$
\begin{array}{lll}
v \models \theta_1 \geq \theta_2 & \text{iff} & [\![\theta_1]\!]_v \geq [\![\theta_2]\!]_v \\
v \models [\alpha]\phi & \text{iff} & w \models \phi \text{ for all } w \text{ with } v\rho(\alpha)w \\
v \models \langle\alpha\rangle\phi & \text{iff} & w \models \phi \text{ for some } w \text{ with } v\rho(\alpha)w \\
v \models \forall x\, \phi & \text{iff} & w \models \phi \text{ for all } w \text{ that agree with } v \text{ except for } x \\
v \models \exists x\, \phi & \text{iff} & w \models \phi \text{ for some } w \text{ that agrees with } v \text{ except for } x \\
v \models \phi \wedge \psi & \text{iff} & v \models \phi \text{ and } v \models \psi
\end{array}
$$

Outline

Differential Dynamic Logic: Axiomatization

equations of truth

$$[:=] \quad [x := \theta]\phi(x) \leftrightarrow \phi(\theta)$$

$$[?] \quad [?H]\phi \leftrightarrow (H \to \phi)$$

$$['] \quad [x' = f(x)]\phi \leftrightarrow \forall t{\geq}0\,[x := y(t)]\phi \qquad (y'(t) = f(y))$$

$$[\cup] \quad [\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \land [\beta]\phi$$

$$[;] \quad [\alpha;\beta]\phi \leftrightarrow [\alpha][\beta]\phi$$

$$[^*] \quad [\alpha^*]\phi \leftrightarrow \phi \land [\alpha][\alpha^*]\phi$$

$$\mathrm{K} \quad [\alpha](\phi \to \psi) \to ([\alpha]\phi \to [\alpha]\psi)$$

$$\mathrm{I} \quad [\alpha^*](\phi \to [\alpha]\phi) \to (\phi \to [\alpha^*]\phi)$$

$$\mathrm{C} \quad [\alpha^*]\forall v{>}0\,(\varphi(v) \to \langle\alpha\rangle\varphi(v-1)) \to \forall v\,(\varphi(v) \to \langle\alpha^*\rangle\exists v{\leq}0\,\varphi(v))$$

LICS'12

Differential Dynamic Logic: Axiomatization

equations of truth

G $\dfrac{\phi}{[\alpha]\phi}$

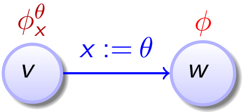
MP $\dfrac{\phi \rightarrow \psi \quad \phi}{\psi}$

$\forall$ $\dfrac{\phi}{\forall x\, \phi}$

LICS'12

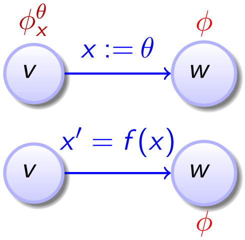
Proofs for Hybrid Systems

$$\frac{\phi_x^\theta}{[x := \theta]\phi}$$



$$\frac{\phi_x^\theta}{[x := \theta]\phi}$$

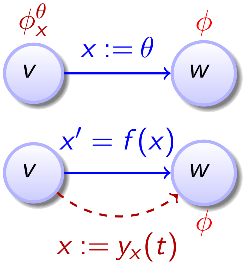
$$\frac{\forall t{\geq}0\, [x := y_x(t)]\phi}{[x' = f(x)]\phi}$$



Proofs for Hybrid Systems

$$\frac{\phi_x^\theta}{[x := \theta]\phi}$$

$$\frac{\forall t{\geq}0\, [x := y_x(t)]\phi}{[x' = f(x)]\phi}$$

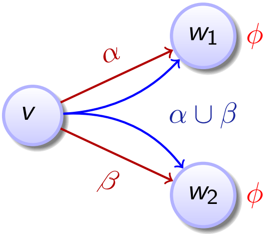


Proofs for Hybrid Systems

compositional semantics $\Rightarrow$ compositional rules!

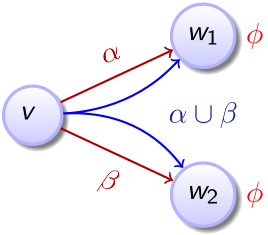
Proofs for Hybrid Systems

$$\frac{[\alpha]\phi \land [\beta]\phi}{[\alpha \cup \beta]\phi}$$

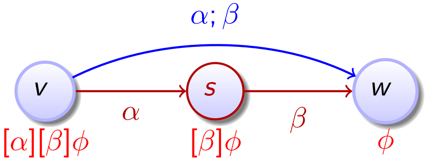


Proofs for Hybrid Systems

$$\frac{[\alpha]\phi \land [\beta]\phi}{[\alpha \cup \beta]\phi}$$

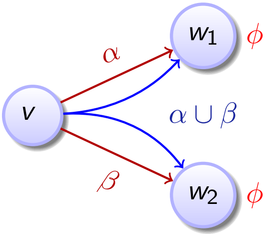


$$\frac{[\alpha][\beta]\phi}{[\alpha; \beta]\phi}$$

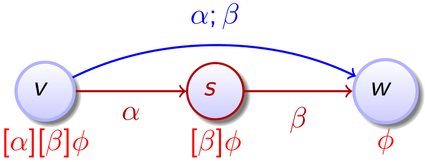


Proofs for Hybrid Systems

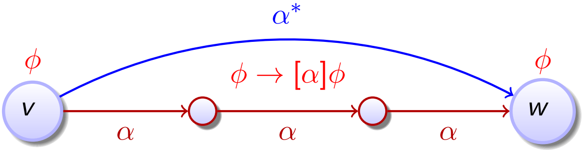
$$\frac{[\alpha]\phi \land [\beta]\phi}{[\alpha \cup \beta]\phi}$$



$$\frac{[\alpha][\beta]\phi}{[\alpha;\beta]\phi}$$



$$\frac{\phi \quad (\phi \to [\alpha]\phi)}{[\alpha^*]\phi}$$



Outline

1. CPS are Multi-Dynamical Systems
   - Hybrid Systems
   - Hybrid Games
2. Dynamic Logic for Multi-Dynamical Systems
   - Syntax
   - Semantics
3. Proofs for CPS
4. Theory of CPS
   - Soundness and Completeness
   - Differential Invariants
   - Differential Radical Invariants
5. Applications
   - Ground Robots
6. Summary

Complete Proof Theory of Hybrid Systems

Theorem (Sound & Complete) (J.Autom.Reas. 2008, LICS'12)

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to either differential equations* **or** *discrete dynamics.* ▸ Proof 25pp

Corollary (Complete Proof-theoretical Alignment & Bridging)

proving continuous = proving hybrid = proving discrete

JAutomReas'08,LICS'12

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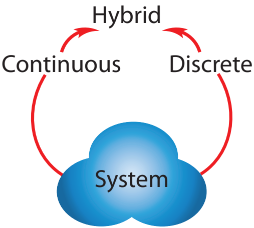

JAutomReas'08,LICS'12

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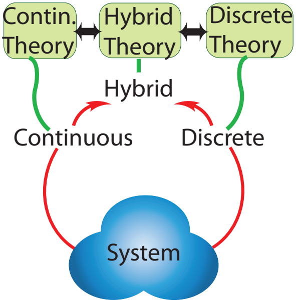
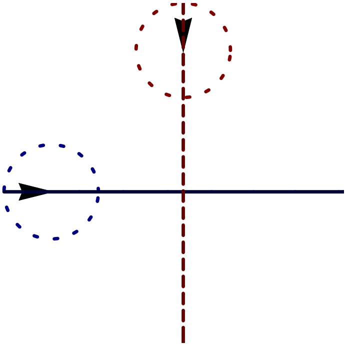
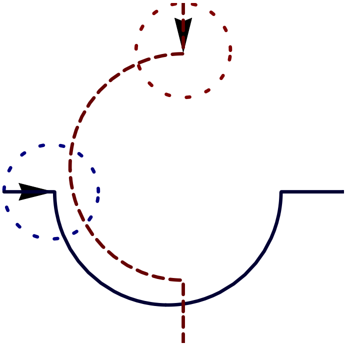


JAutomReas'08,LICS'12

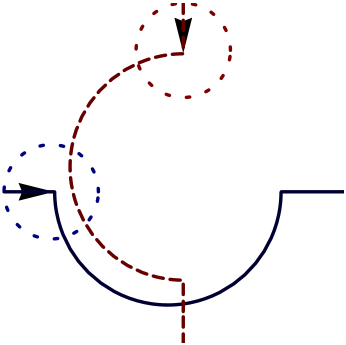
Air Traffic Control



Air Traffic Control



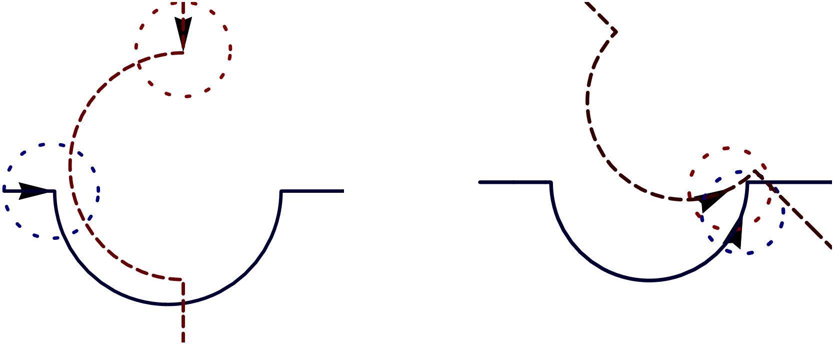
Air Traffic Control



Verification?

looks correct

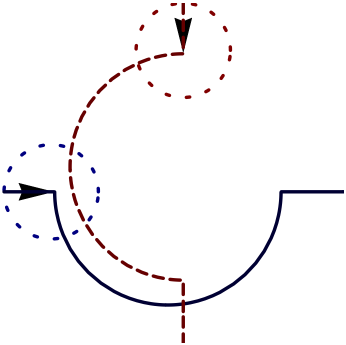
Air Traffic Control

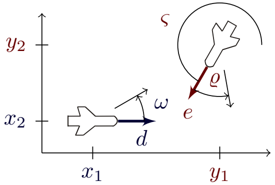


Verification?

looks correct NO!

Air Traffic Control



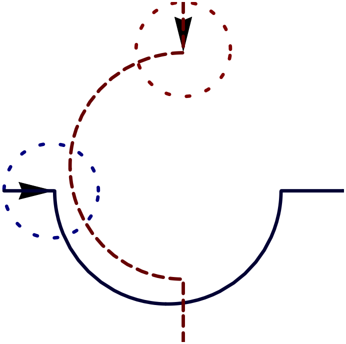


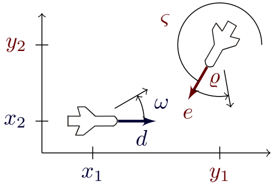
$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Verification?

looks correct NO!

Air Traffic Control

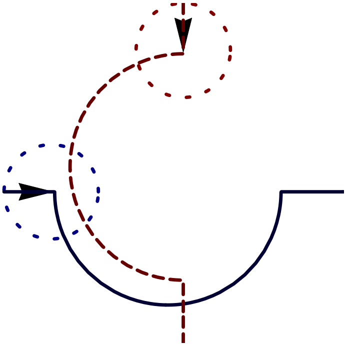


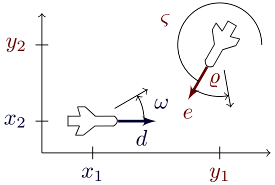


$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Example (“Solving” differential equations)

$$x_1(t) = \frac{1}{\omega\varpi}\big(x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega$$
$$+ x_2\omega\varpi\sin t\omega - v_2\omega\cos\vartheta\cos t\varpi\sin t\omega - v_2\omega\sqrt{1 - \sin\vartheta^2}\sin t\omega$$
$$+ v_2\omega\cos\vartheta\cos t\omega\sin t\varpi + v_2\omega\sin\vartheta\sin t\omega\sin t\varpi\big)\ldots$$





$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Example ("Solving" differential equations)

$$\forall t \geq 0 \quad \frac{1}{\omega\varpi}\Big(x_1\omega\varpi \cos t\omega - v_2\omega \cos t\omega \sin\vartheta + v_2\omega \cos t\omega \cos t\varpi \sin\vartheta - v_1\varpi \sin t\omega$$
$$+ x_2\omega\varpi \sin t\omega - v_2\omega \cos\vartheta \cos t\varpi \sin t\omega - v_2\omega\sqrt{1 - \sin\vartheta^2} \sin t\omega$$
$$+ v_2\omega \cos\vartheta \cos t\omega \sin t\varpi + v_2\omega \sin\vartheta \sin t\omega \sin t\varpi\Big) \ldots$$

```
\forall R ts2.
  (   0 <= ts2 & ts2 <= t2_0
   ->       (  (om_1)^-1
             * (omb_1)^-1
             * (  om_1 * omb_1 * x1 * Cos(om_1 * ts2)
                + om_1 * v2 * Cos(om_1 * ts2) * (1 + -1 * (Cos(u))^2)^(1 / 2)
                + -1 * omb_1 * v1 * Sin(om_1 * ts2)
                + om_1 * omb_1 * x2 * Sin(om_1 * ts2)
                + om_1 * v2 * Cos(u) * Sin(om_1 * ts2)
                + -1 * om_1 * v2 * Cos(omb_1 * ts2) * Cos(u) * Sin(om_1 * ts2)
                + om_1 * v2 * Cos(om_1 * ts2) * Cos(u) * Sin(omb_1 * ts2)
                + om_1 * v2 * Cos(om_1 * ts2) * Cos(omb_1 * ts2) * Sin(u)
                + om_1 * v2 * Sin(om_1 * ts2) * Sin(omb_1 * ts2) * Sin(u)))
        ^2
       +    (  (om_1)^-1
             * (omb_1)^-1
             * (  -1 * omb_1 * v1 * Cos(om_1 * ts2)
                + om_1 * omb_1 * x2 * Cos(om_1 * ts2)
                + omb_1 * v1 * (Cos(om_1 * ts2))^2
                + om_1 * v2 * Cos(om_1 * ts2) * Cos(u)
                + -1 * om_1 * v2 * Cos(om_1 * ts2) * Cos(omb_1 * ts2) * Cos(u)
                + -1 * om_1 * omb_1 * x1 * Sin(om_1 * ts2)
                +   -1
                  * om_1
                  * v2
                  * (1 + -1 * (Cos(u))^2)^(1 / 2)
                  * Sin(om_1 * ts2)
                + omb_1 * v1 * (Sin(om_1 * ts2))^2
                + -1 * om_1 * v2 * Cos(u) * Sin(om_1 * ts2) * Sin(omb_1 * ts2)
                + -1 * om_1 * v2 * Cos(omb_1 * ts2) * Sin(om_1 * ts2) * Sin(u)
                + om_1 * v2 * Cos(om_1 * ts2) * Sin(omb_1 * ts2) * Sin(u)))
        ^2
      >= (p)^2),
 t2_0 >= 0,
 x1^2 + x2^2 >= (p)^2
==>
```

```
\forall R t7.
  (   t7 >= 0
   ->       (   (om_3)^-1
             * (    om_3
                 * (   (om_1)^-1
                    * (omb_1)^-1
                    * (   om_1 * omb_1 * x1 * Cos(om_1 * t2_0)
                       +   om_1
                         * v2
                         * Cos(om_1 * t2_0)
                         * (1 + -1 * (Cos(u))^2)^(1 / 2)
                       + -1 * omb_1 * v1 * Sin(om_1 * t2_0)
                       + om_1 * omb_1 * x2 * Sin(om_1 * t2_0)
                       + om_1 * v2 * Cos(u) * Sin(om_1 * t2_0)
                       +   -1
                         * om_1
                         * v2
                         * Cos(omb_1 * t2_0)
                         * Cos(u)
                         * Sin(om_1 * t2_0)
                       +   om_1
                         * v2
                         * Cos(om_1 * t2_0)
                         * Cos(u)
                         * Sin(omb_1 * t2_0)
                       +   om_1
                         * v2
                         * Cos(om_1 * t2_0)
                         * Cos(omb_1 * t2_0)
                         * Sin(u)
                       +   om_1
                         * v2
                         * Sin(om_1 * t2_0)
                         * Sin(omb_1 * t2_0)
                         * Sin(u)))
```

```
    * Cos(om_3 * t5)
  +   v2
    * Cos(om_3 * t5)
    *   (  1
         +   -1
           * (Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4))^2)
     ^(1 / 2)
  + -1 * v1 * Sin(om_3 * t5)
  +   om_3
    * (  (om_1)^-1
       * (omb_1)^-1
       * (  -1 * omb_1 * v1 * Cos(om_1 * t2_0)
          + om_1 * omb_1 * x2 * Cos(om_1 * t2_0)
          + omb_1 * v1 * (Cos(om_1 * t2_0))^2
          + om_1 * v2 * Cos(om_1 * t2_0) * Cos(u)
          +   -1
            * om_1
            * v2
            * Cos(om_1 * t2_0)
            * Cos(omb_1 * t2_0)
            * Cos(u)
          + -1 * om_1 * omb_1 * x1 * Sin(om_1 * t2_0)
          +   -1
            * om_1
            * v2
            * (1 + -1 * (Cos(u))^2)^(1 / 2)
            * Sin(om_1 * t2_0)
          + omb_1 * v1 * (Sin(om_1 * t2_0))^2
          +   -1
            * om_1
            * v2
            * Cos(u)
            * Sin(om_1 * t2_0)
            * Sin(omb_1 * t2_0)
```

```
                  +   -1
                    * om_1
                    * v2
                    * Cos(omb_1 * t2_0)
                    * Sin(om_1 * t2_0)
                    * Sin(u)
                  +   om_1
                    * v2
                    * Cos(om_1 * t2_0)
                    * Sin(omb_1 * t2_0)
                    * Sin(u)))
            * Sin(om_3 * t5)
          +   v2
            * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
            * Sin(om_3 * t5)
          +   v2
            * (Cos(om_3 * t5))^2
            * Sin(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
          +   v2
            * (Sin(om_3 * t5))^2
            * Sin(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)))
 ^2
+   (  (om_3)^-1
     * (  -1 * v1 * Cos(om_3 * t5)
        +   om_3
          * (  (om_1)^-1
             * (omb_1)^-1
             * (  -1 * omb_1 * v1 * Cos(om_1 * t2_0)
                + om_1 * omb_1 * x2 * Cos(om_1 * t2_0)
                + omb_1 * v1 * (Cos(om_1 * t2_0))^2
                + om_1 * v2 * Cos(om_1 * t2_0) * Cos(u)
                +   -1
                  * om_1
                  * v2
                  * Cos(om_1 * t2_0)
                  * Cos(omb_1 * t2_0)
```

```
        + -1 * om_1 * omb_1 * x1 * Sin(om_1 * t2_0)
        +   -1
          * om_1
          * v2
          * (1 + -1 * (Cos(u))^2)^(1 / 2)
          * Sin(om_1 * t2_0)
        + omb_1 * v1 * (Sin(om_1 * t2_0))^2
        +   -1
          * om_1
          * v2
          * Cos(u)
          * Sin(om_1 * t2_0)
          * Sin(omb_1 * t2_0)
        +   -1
          * om_1
          * v2
          * Cos(omb_1 * t2_0)
          * Sin(om_1 * t2_0)
          * Sin(u)
        +   om_1
          * v2
          * Cos(om_1 * t2_0)
          * Sin(omb_1 * t2_0)
          * Sin(u)))
  * Cos(om_3 * t5)
+ v1 * (Cos(om_3 * t5))^2
+   v2
  * Cos(om_3 * t5)
  * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
+   -1
  * v2
  * (Cos(om_3 * t5))^2
  * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
```

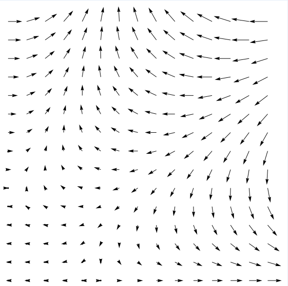
```
+   -1
  * om_3
  * (  (om_1)^-1
     * (omb_1)^-1
     * (  om_1 * omb_1 * x1 * Cos(om_1 * t2_0)
        +   om_1
          * v2
          * Cos(om_1 * t2_0)
          * (1 + -1 * (Cos(u))^2)^(1 / 2)
        + -1 * omb_1 * v1 * Sin(om_1 * t2_0)
        + om_1 * omb_1 * x2 * Sin(om_1 * t2_0)
        + om_1 * v2 * Cos(u) * Sin(om_1 * t2_0)
        +   -1
          * om_1
          * v2
          * Cos(omb_1 * t2_0)
          * Cos(u)
          * Sin(om_1 * t2_0)
        +   om_1
          * v2
          * Cos(om_1 * t2_0)
          * Cos(u)
          * Sin(omb_1 * t2_0)
        +   om_1
          * v2
          * Cos(om_1 * t2_0)
          * Cos(omb_1 * t2_0)
          * Sin(u)
        +   om_1
          * v2
          * Sin(om_1 * t2_0)
          * Sin(omb_1 * t2_0)
          * Sin(u)))
  * Sin(om_3 * t5)
```

```
        +   -1
          * v2
          *   (  1
                +   -1
                  * (Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4))^2)
           ^(1 / 2)
          * Sin(om_3 * t5)
        + v1 * (Sin(om_3 * t5))^2
        +   -1
          * v2
          * Cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
          * (Sin(om_3 * t5))^2))
  ^2
>= (p)^2)
```

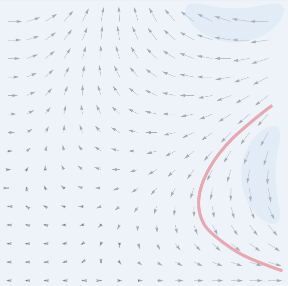
This is just one branch to prove for aircraft ...

Differential Invariants for Differential Equations

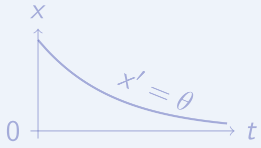
Differential Invariant

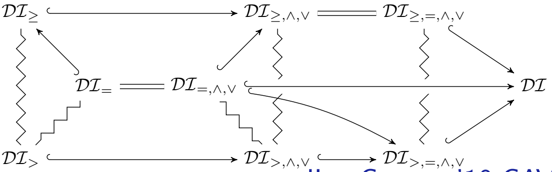


Differential Cut



Differential Ghost





Logic

Provability study

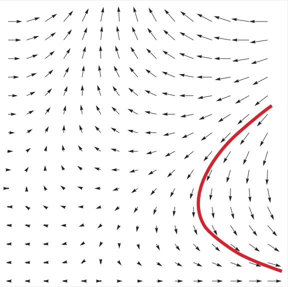
Math

Character- istic PDE

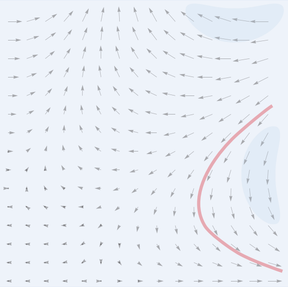
JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

Differential Invariants for Differential Equations

Differential Invariant



Differential Cut



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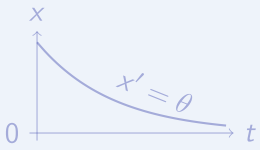

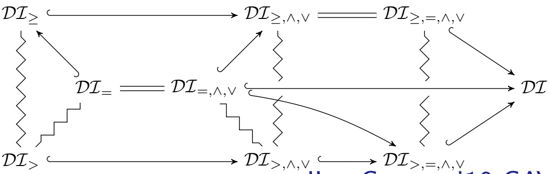

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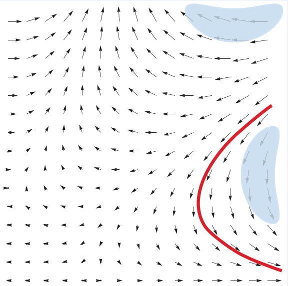
Differential Invariant



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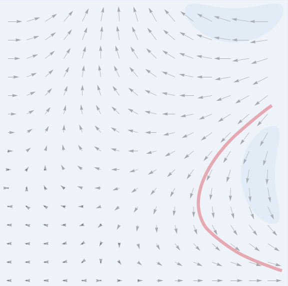

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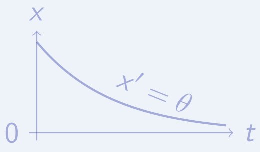

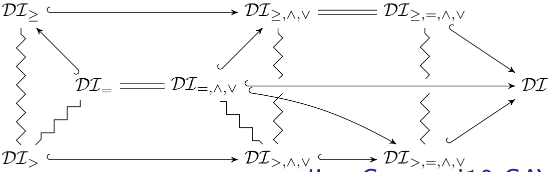

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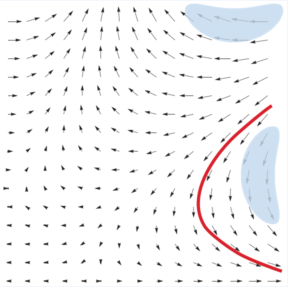
Differential Invariants for Differential Equations

Differential Invariant



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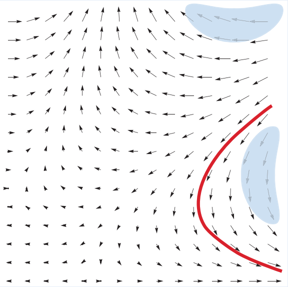

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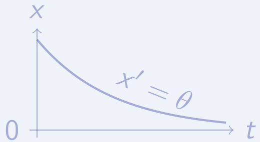

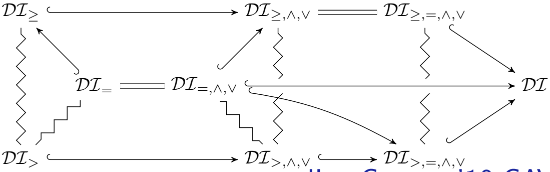

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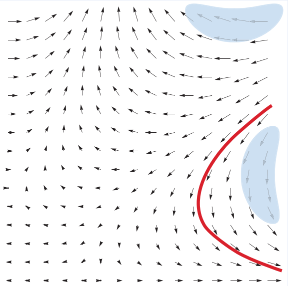
Differential Invariants for Differential Equations

Differential Invariant



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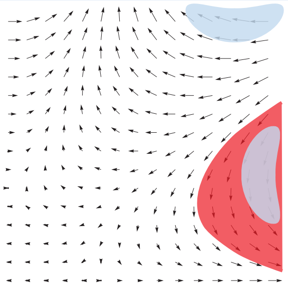

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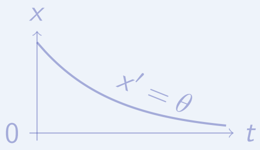

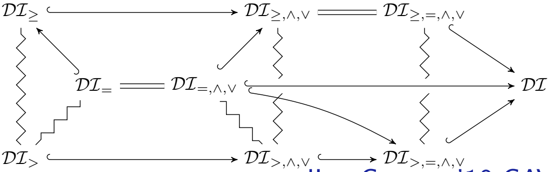

Logic

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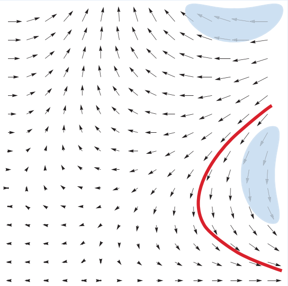
Differential Invariant



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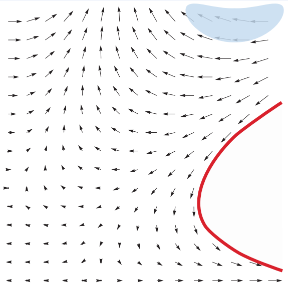

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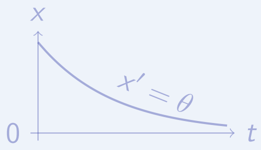

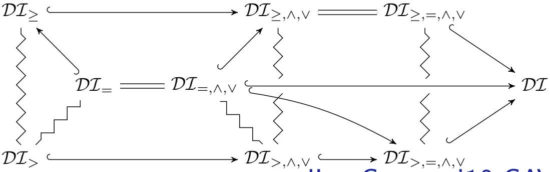

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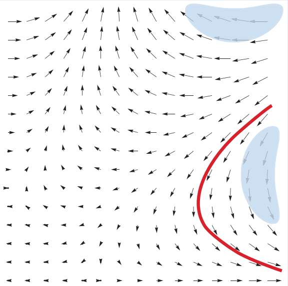
Differential Invariants for Differential Equations

Differential Invariant



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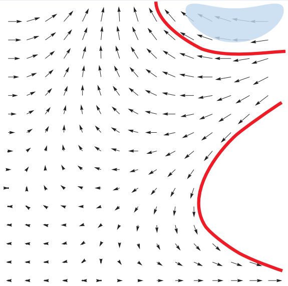

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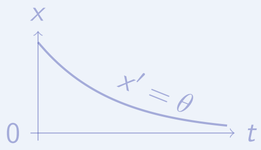

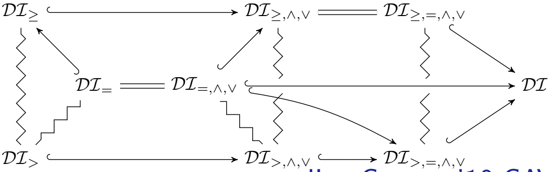

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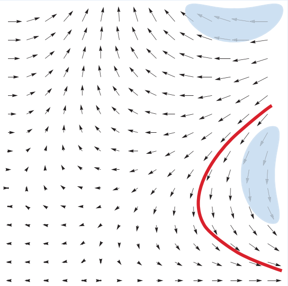
Differential Invariants for Differential Equations

Differential Invariant



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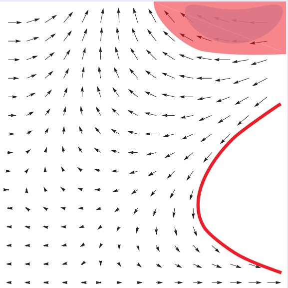

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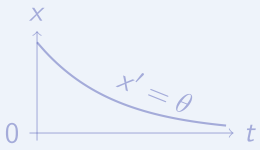

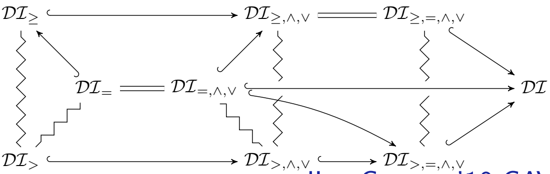

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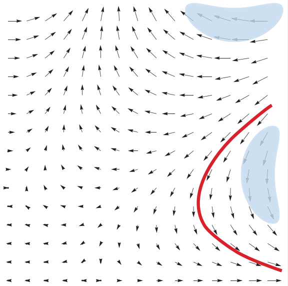
Differential Invariants for Differential Equations

Differential Invariant



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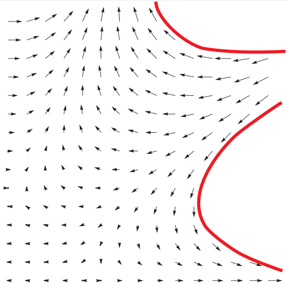

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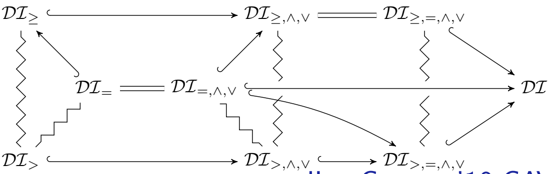

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Differential Invariants for Differential Equations

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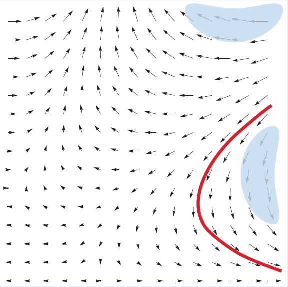

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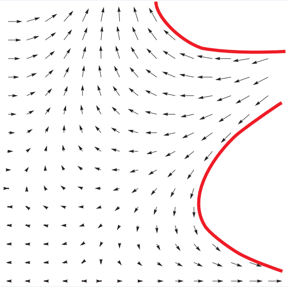

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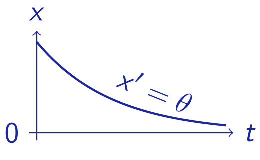

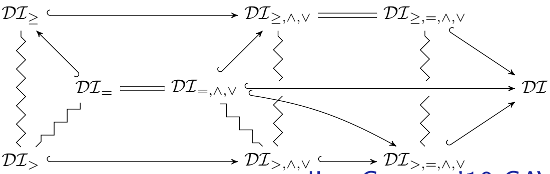

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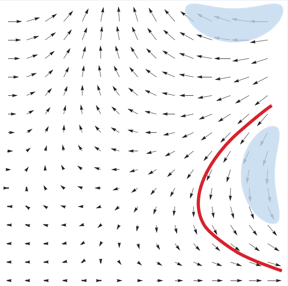
Differential Invariants for Differential Equations

Differential Invariant



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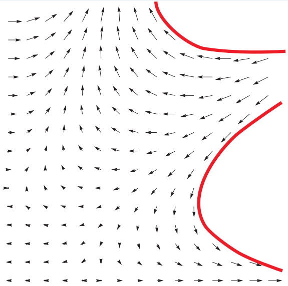

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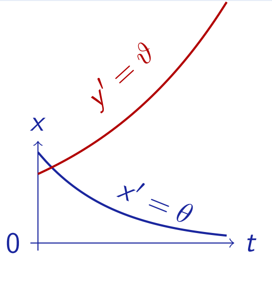

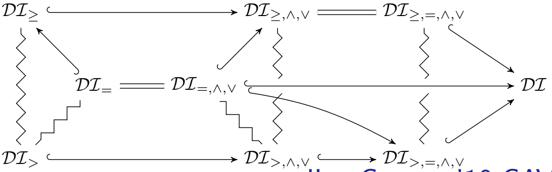

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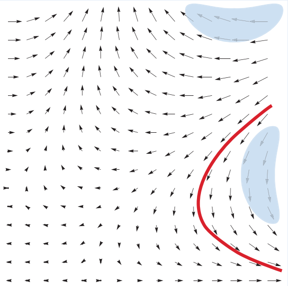
Differential Invariants for Differential Equations

Differential Invariant



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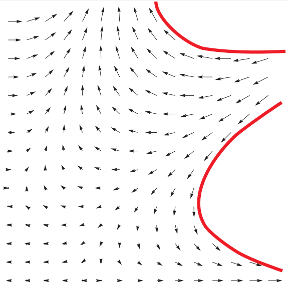

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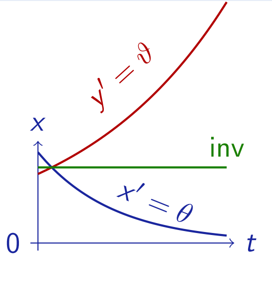

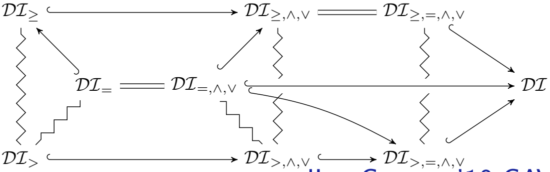

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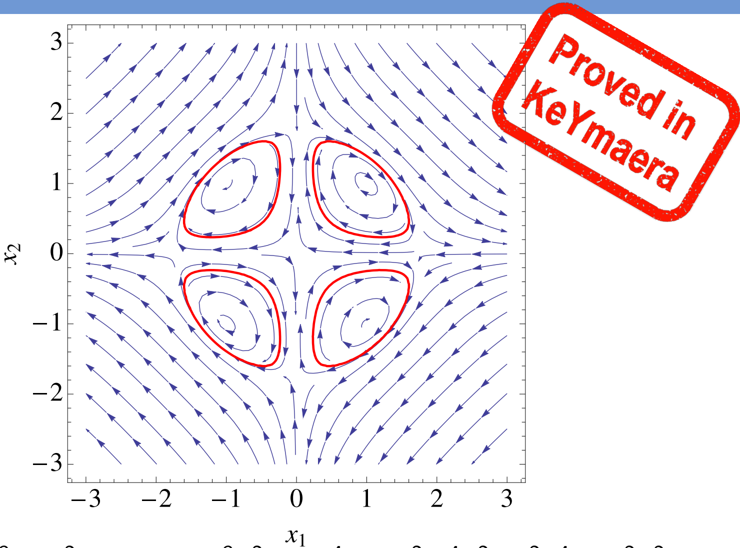
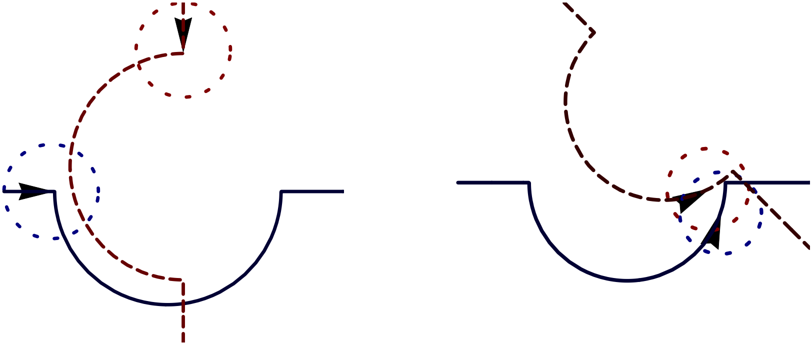


$$[x' = 2x^4y+4x^2y^3-6x^2y, y' = -4x^3y^2-2xy^4+6xy^2]x^4y^2+x^2y^4-3x^2y^2 \leq c$$

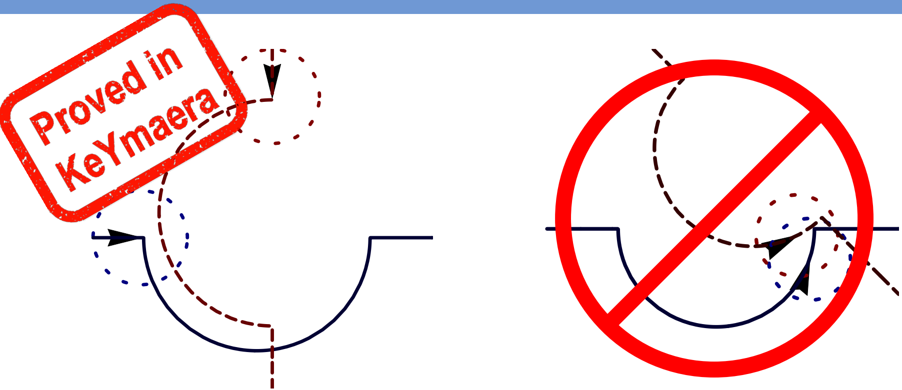
Air Traffic Control



Verification

Simple proof distinguishes safe from unsafe flight maneuver

Air Traffic Control



Verification

Simple proof distinguishes safe from unsafe flight maneuver

Differential Radical Invariants

Theorem (Differential radical invariant characterization)

$$\frac{h = 0 \rightarrow \bigwedge_{i=0}^{N-1} \mathcal{L}_p^{(i)}(h) = 0}{h = 0 \rightarrow [x' = p]h = 0}$$

characterizes all algebraic invariants, where $N = ord\ \sqrt[\mathcal{L}]{(h)}$*, i.e.*

$$\mathcal{L}_p^{(N)}(h) = \sum_{i=0}^{N-1} g_i \mathcal{L}_p^{(i)}(h)$$

Corollary (Algebraic Invariants Decidable)

Invariance decidable for real algebraic $h = 0$*.*

TACAS'14

Case Study: Longitudinal Dynamics of an Airplane

6th Order Longitudinal Equations

$$u' = \frac{X}{m} - g\sin(\theta) - qw \qquad u : \text{axial velocity}$$

$$w' = \frac{Z}{m} + g\cos(\theta) + qu \qquad w : \text{vertical velocity}$$

$$x' = \cos(\theta)u + \sin(\theta)w \qquad x : \text{range}$$

$$z' = -\sin(\theta)u + \cos(\theta)w \qquad z : \text{altitude}$$

$$\theta' = q \qquad \theta : \text{pitch angle}$$

$$q' = \frac{M}{I_{yy}} \qquad q : \text{pitch rate}$$

X : thrust along u, Z : thrust along w, M : thrust moment for w
g : gravity, m : mass, I_{yy} : inertia second diagonal

Case Study: Automatically Generated Invariants

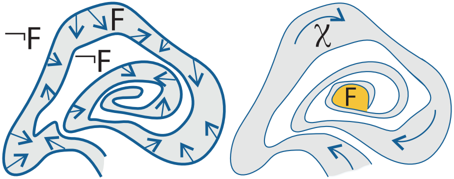
Automatically Generated Invariant Functions

$$\frac{Mz}{I_{yy}} + g\theta + \left(\frac{X}{m} - qw\right)\cos(\theta) + \left(\frac{Z}{m} + qu\right)\sin(\theta)$$

$$\frac{Mx}{I_{yy}} - \left(\frac{Z}{m} + qu\right)\cos(\theta) + \left(\frac{X}{m} - qw\right)\sin(\theta)$$

$$-q^2 + \frac{2M\theta}{I_{yy}}$$

Outline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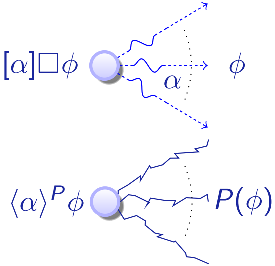
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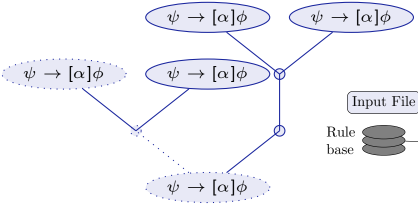
$P(\phi)$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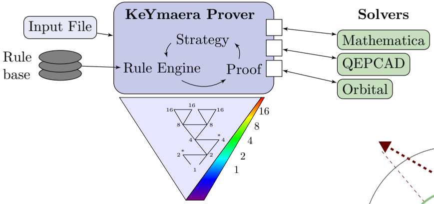
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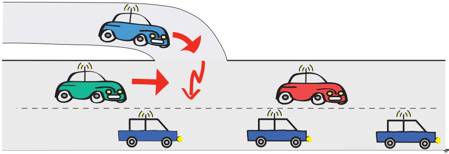

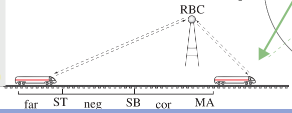
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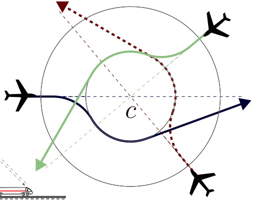
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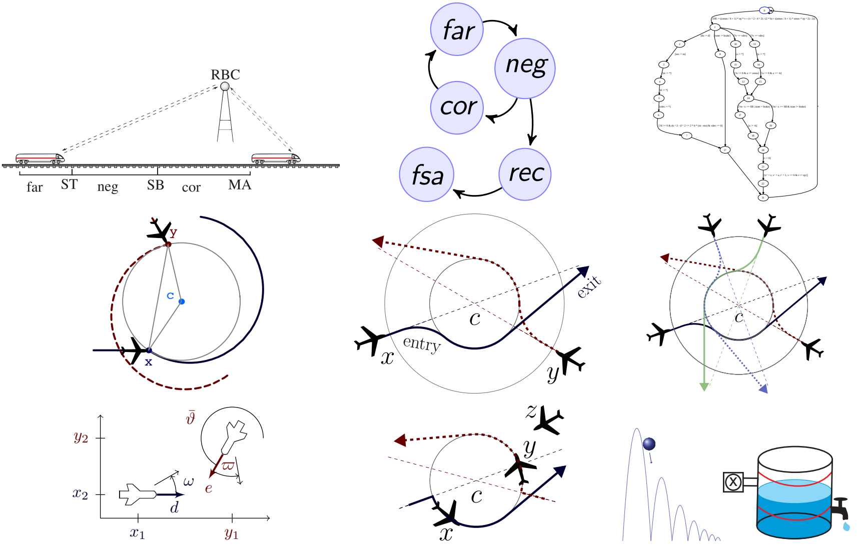
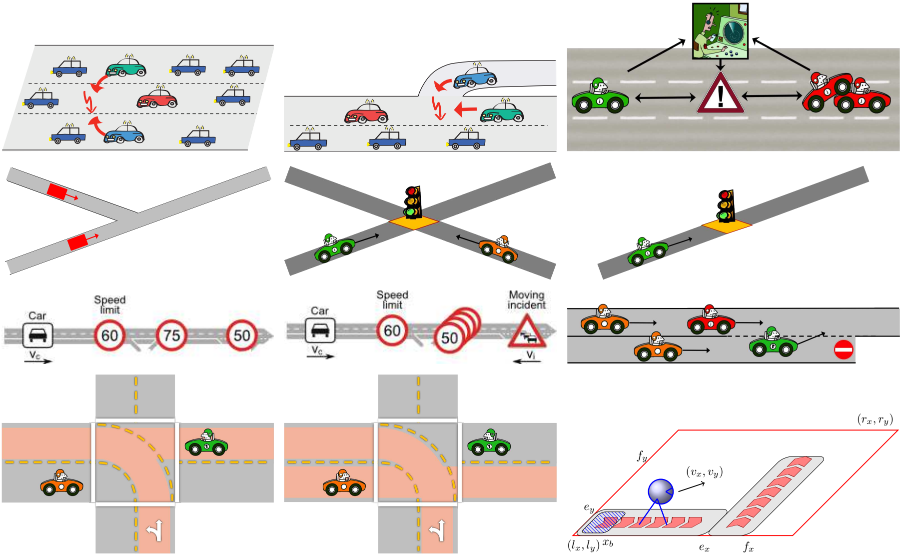
Successful CPS Proofs



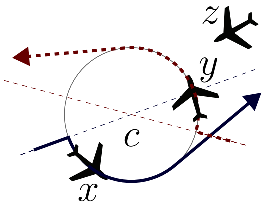
ICFEM'09,CAV'08,FM'09,HSCC'11

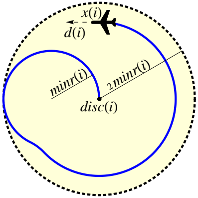
Successful CPS Proofs

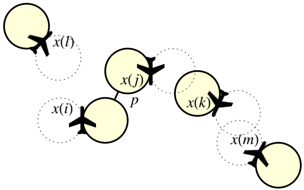


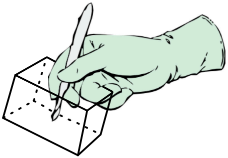
FM'11,LMCS'12,ICCPS'12,ITSC'11,ITSC'13,IJCAR'12

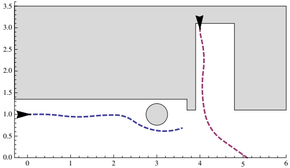
Successful CPS Proofs

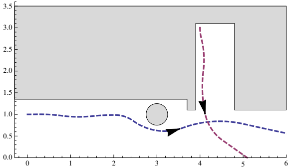


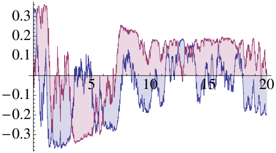


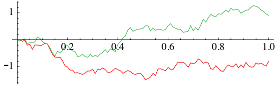


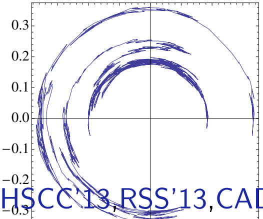




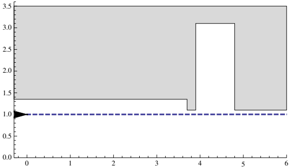
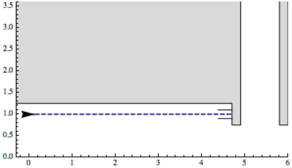
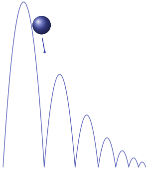
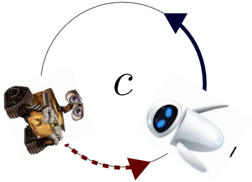


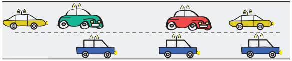
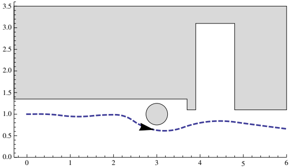
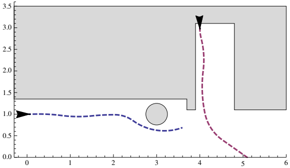
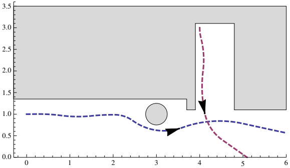






HSCC'11,HSCC'13,HSCC'13,RSS'13,CADE'12

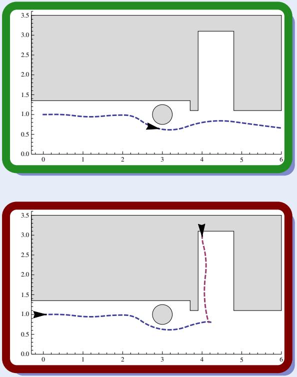





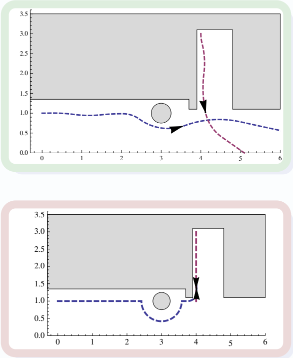
students in 15-424/624 *Foundations of Cyber-Physical Systems* course

What is Safe?

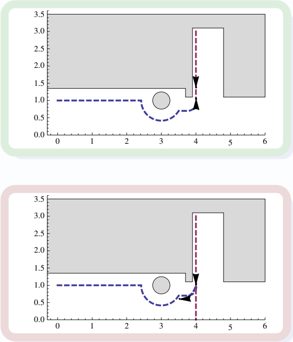
Static safety



Passive safety



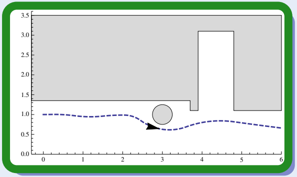
Passive friendly

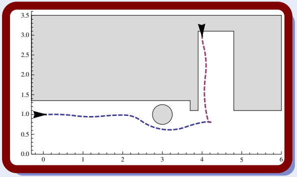


✓ Verified with KeYmaera

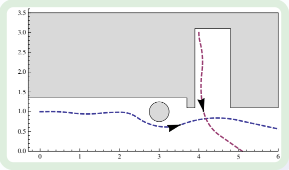
Sensor failure   Actuator disturbance   Robot and obstacle shape

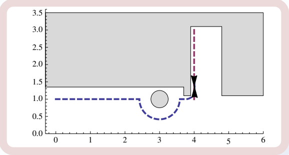
Static safety



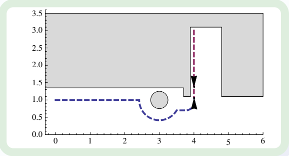


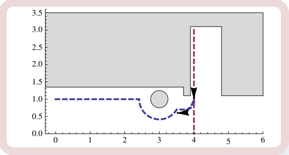
Passive safety





Passive friendly





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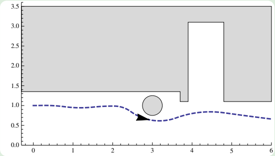
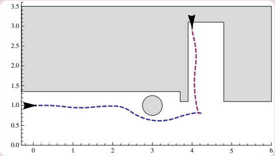
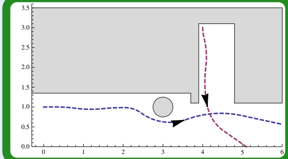
Sensor failure   Actuator disturbance   Robot and obstacle shape

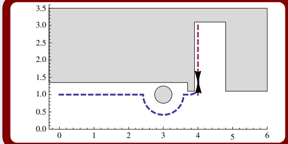
Static safety



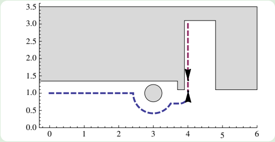


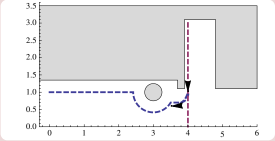
Passive safety





Passive friendly





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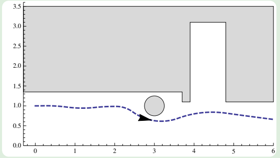
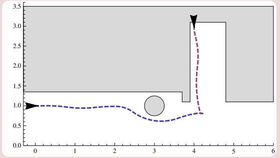
▸ Sensor failure ▸ Actuator disturbance ▸ Robot and obstacle shape

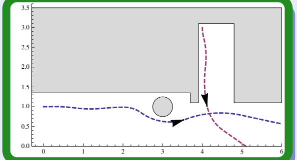
What is Safe?

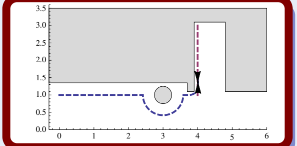
Static safety





Passive safety





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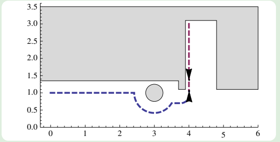

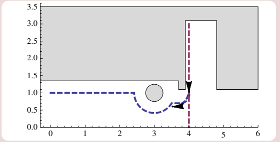

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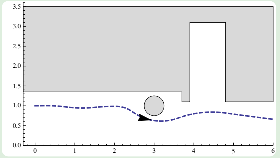
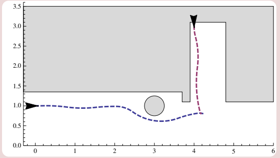
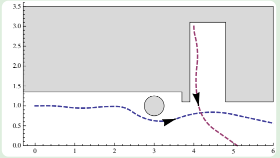
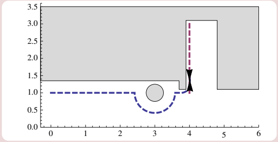
Static safety





Passive safety





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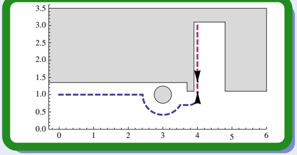

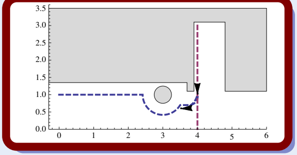

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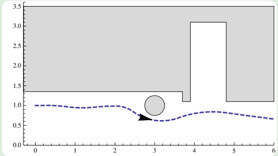
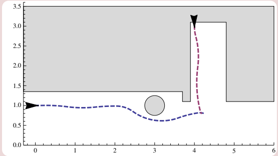
What is Safe?

Static safety





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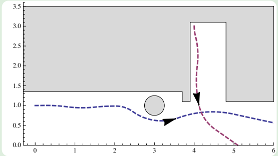

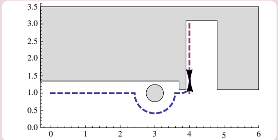

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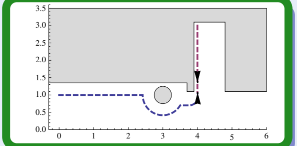
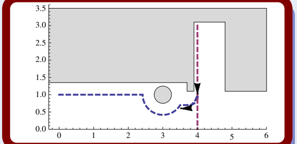


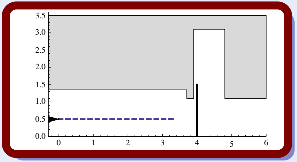


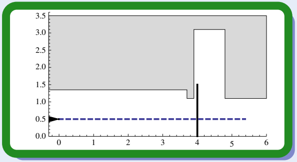
✓ Verified with KeYmaera

Sensor failure · Actuator disturbance · Robot and obstacle shape

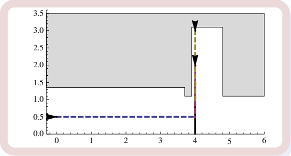
What is the Goal of the Robot?

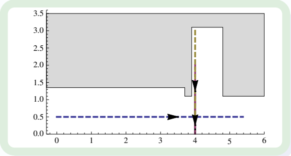
Pass waypoint



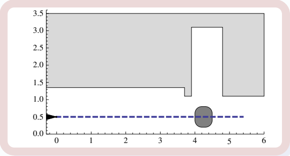


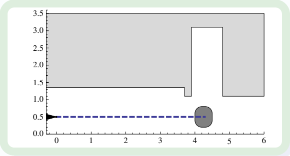
Cross intersection





Reach area

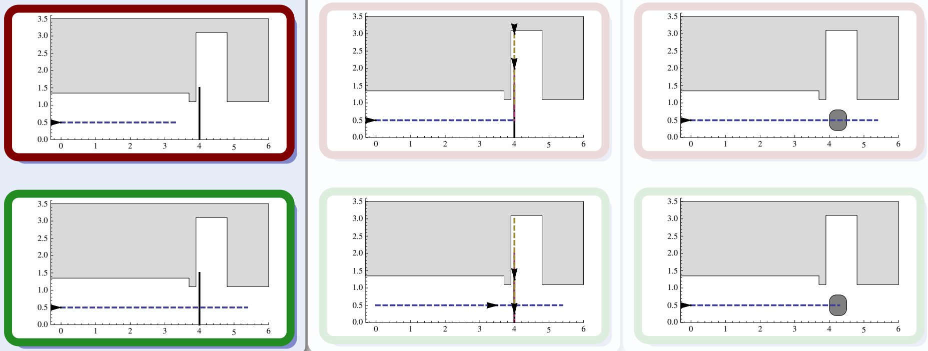




✓ Verified with KeYmaera

What is the Goal of the Robot?

Pass waypoint | Cross intersection | Reach area



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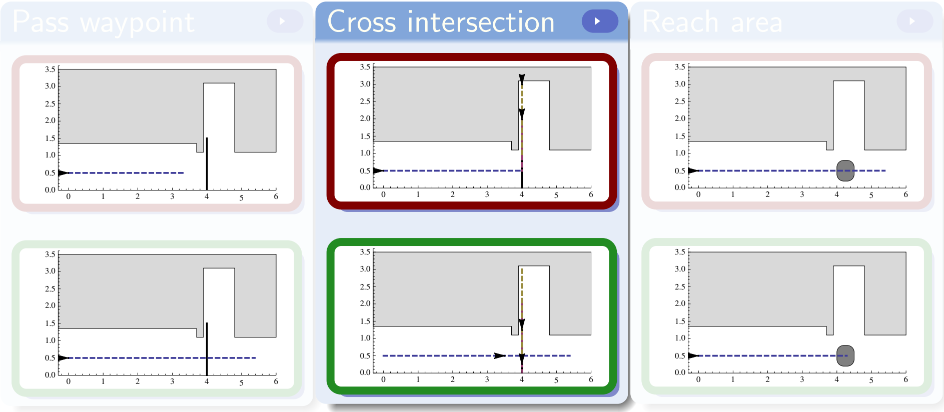
What is the Goal of the Robot?



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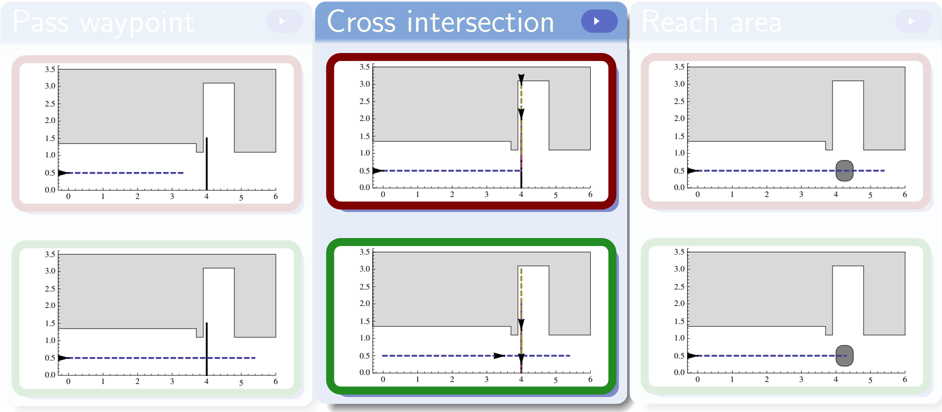

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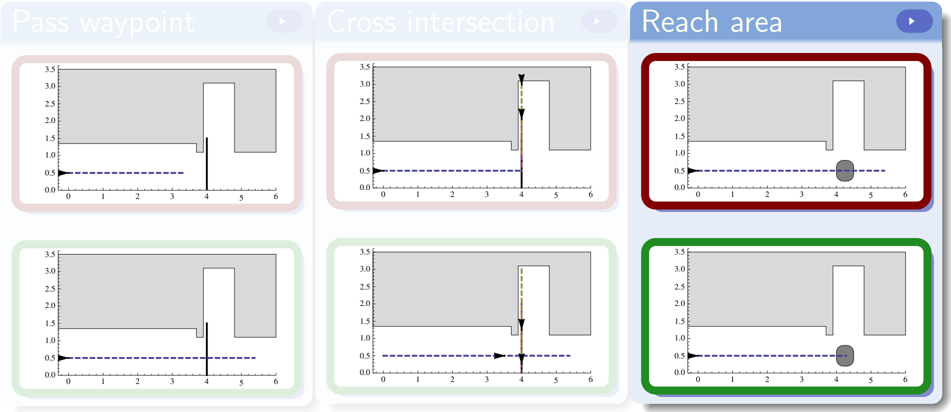

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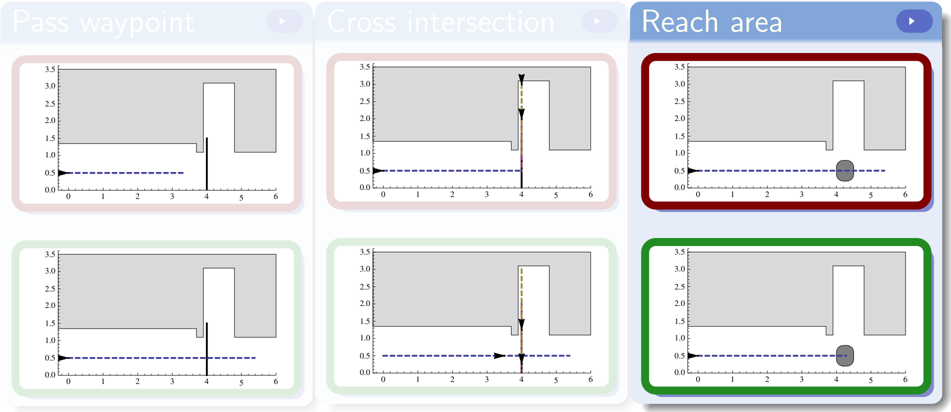


✓ Verified with KeYmaera

Robot Invariants and Constraints

Safety	Invariant + Safe Control (RSS'13)
static	$\lVert p_r - p_o \rVert_\infty > \frac{v_r^2}{2b} + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon v_r\right)$
passive	$v_r = 0 \lor \lVert p_r - p_o \rVert_\infty > \frac{v_r^2}{2b} + V\frac{v_r}{b} + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon(v_r + V)\right)$
+ sensor	$\lVert \hat{p}_r - p_o \rVert_\infty > \frac{v_r^2}{2b} + V\frac{v_r}{b} + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon(v_r + V)\right) + U_p$
+ disturb	$\lVert p_r - p_o \rVert_\infty > \frac{v_r^2}{2bU_m} + V\frac{v_r}{bU_m} + \left(\frac{A}{bU_m}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon(v_r + V)\right)$
+ failure	$\lVert \hat{p}_r - p_o \rVert_\infty > \frac{v_r^2}{2b} + V\frac{v_r}{b} + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon(v + V)\right) + U_p + g\Delta$
friendly	$\lVert p_r - p_o \rVert_\infty > \frac{v_r^2}{2b} + \frac{V^2}{2b_o} + V\left(\frac{v_r}{b} + \tau\right) + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon(v_r + V)\right)$

RSS'13

Robot Invariants and Constraints

Safety	Invariant + Safe Control	(RSS'13)
static	$\|p_r - p_o\|_\infty > \frac{v_r^2}{2b} + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon v_r\right)$	
passive	$v_r = 0 \lor \|p_r - p_o\|_\infty > \frac{v_r^2}{2b} + V\frac{v_r}{b} + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon(v_r + V)\right)$	
+ sensor	$\ldots + \varepsilon(v_r + V)\big) + U_p$	
+ disturb	$\|p_r - p_o\|_\infty > \frac{v_r^2}{2bU_m} + V\frac{v_r}{bU_m} + \left(\frac{A}{bU_m}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon(v_r + V)\right)$	
+ failure	$\|\hat{p}_r - p_o\|_\infty > \frac{v_r^2}{2b} + V\frac{v_r}{b} + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon(v + V)\right) + U_p + g\Delta$	
friendly	$\|p_r - p_o\|_\infty > \frac{v_r^2}{2b} + \frac{V^2}{2b_o} + V\left(\frac{v_r}{b} + \tau\right) + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon(v_r + V)\right)$	

Question

How to find and justify constraints? Proof!

RSS'13

Outline

Benefits of Logical Foundations for V & V

Proving

Safety	Formalize system properties: What is "Safe"? "Reach goal"?
Models	Formalize system models
Assumptions	Make assumptions explicit
Constraints	Reveal invariants, switching conditions, starting conditions
Design	Invariants guide safe controller design
Constructive	Construct models along with their proof

Byproducts

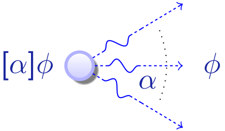
Analyze	Determine design trade-offs & feasibility early
Synthesize	Turn high-level models into code & correctness monitors
Certify	Proofs as artifacts for certification

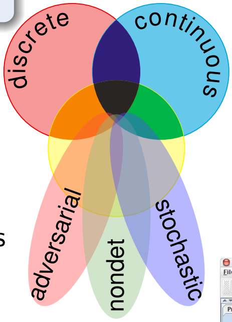
Tools

KeYmaera	Theorem prover for CPS

differential dynamic logic

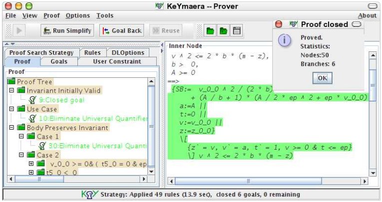
$$d\mathcal{L} = \mathrm{DL} + \mathrm{HP}$$

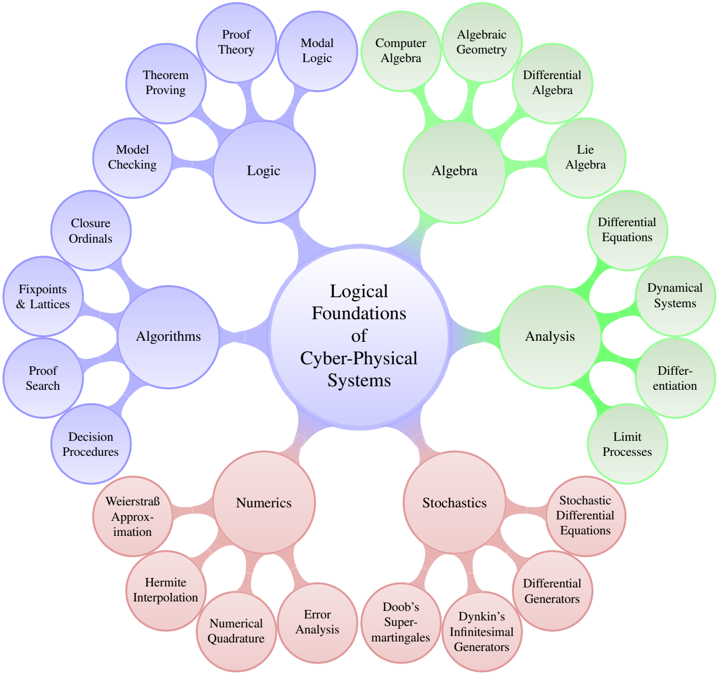


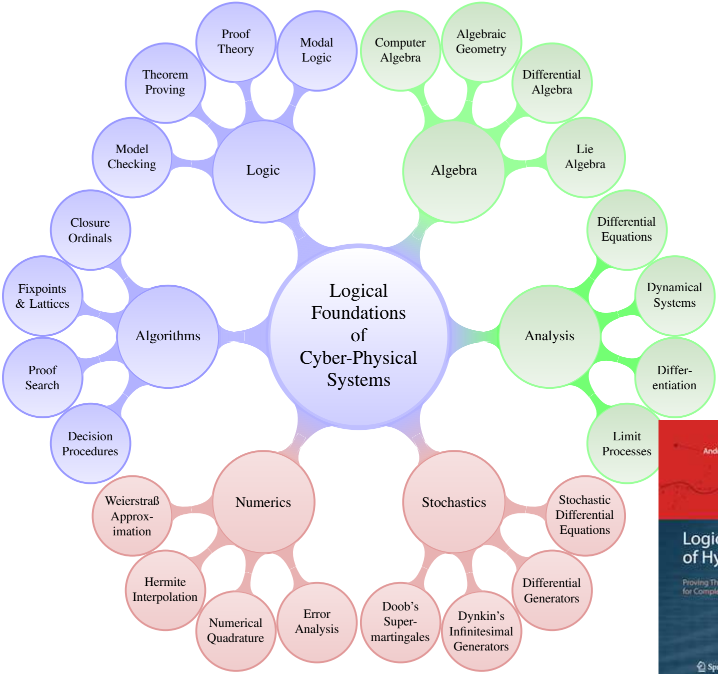


- Multi-dynamical systems
- Combine simple dynamics
- Tame complexity
- Logic & proofs for CPS
- Theory of CPS
- Applications
- Undergrad course 15-424

KeYmaera




Logical Foundations of Cyber-Physical Systems
Logic
Proof Theory
Modal Logic
Theorem Proving
Model Checking
Algebra
Computer Algebra
Algebraic Geometry
Differential Algebra
Lie Algebra
Analysis
Differential Equations
Dynamical Systems
Differ- entiation
Limit Processes
Algorithms
Closure Ordinals
Fixpoints & Lattices
Proof Search
Decision Procedures
Numerics
Weierstraß Approx- imation
Hermite Interpolation
Numerical Quadrature
Error Analysis
Stochastics
Stochastic Differential Equations
Differential Generators
Dynkin's Infinitesimal Generators
Doob's Super- martingales


Logical Foundations of Cyber-Physical Systems
Logic
Proof Theory
Modal Logic
Theorem Proving
Model Checking
Algebra
Computer Algebra
Algebraic Geometry
Differential Algebra
Lie Algebra
Algorithms
Closure Ordinals
Fixpoints & Lattices
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Dynamical Systems
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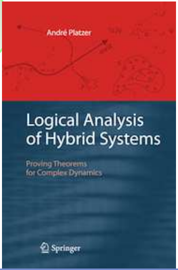
Numerics
Weierstraß Approx-imation
Hermite Interpolation
Numerical Quadrature
Error Analysis
Stochastics
Stochastic Differential Equations
Differential Generators
Dynkin's Infinitesimal Generators
Doob's Super-martingales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